Clifford Geometric Algebra (GA)

A unified mathematical formalism for science and engineering

Vector cross product
$a \times b$

Quaternions q

Complex numbers

Tensors F_{ij}

Spinors

Complex wave functions

Rotation matrices R

Differential forms

“If the world was complicated, everyone would understand i

Woody Allen

A unified language for science

Appropriate mathematical modelling of physical space

~~Quaternions q~~

~~Complex numbers~~

Divide by vectors

Quantum mechanics over the reals

~~Vector cross product a x b~~

GA

Special relativity without an extra time dimension

~~Complex wave functions~~

Strictly real field

Naturally incorporates complex numbers and quaternions

~~Differential forms~~

~~Tensors F_{ij}~~

~~Rotation matrices R~~

~~Spinors~~

Maxwell's four equations reduce to one

Timeline of Mathematics

2000

1637 Cartesian coordinates-Descartes

1545 number line developed

1170-1250 debts seen as negative numbers-Pisa

1000

800 zero used in India

0

BC: negative numbers used in India and China

Euclid’s textbook current for 2000 years!

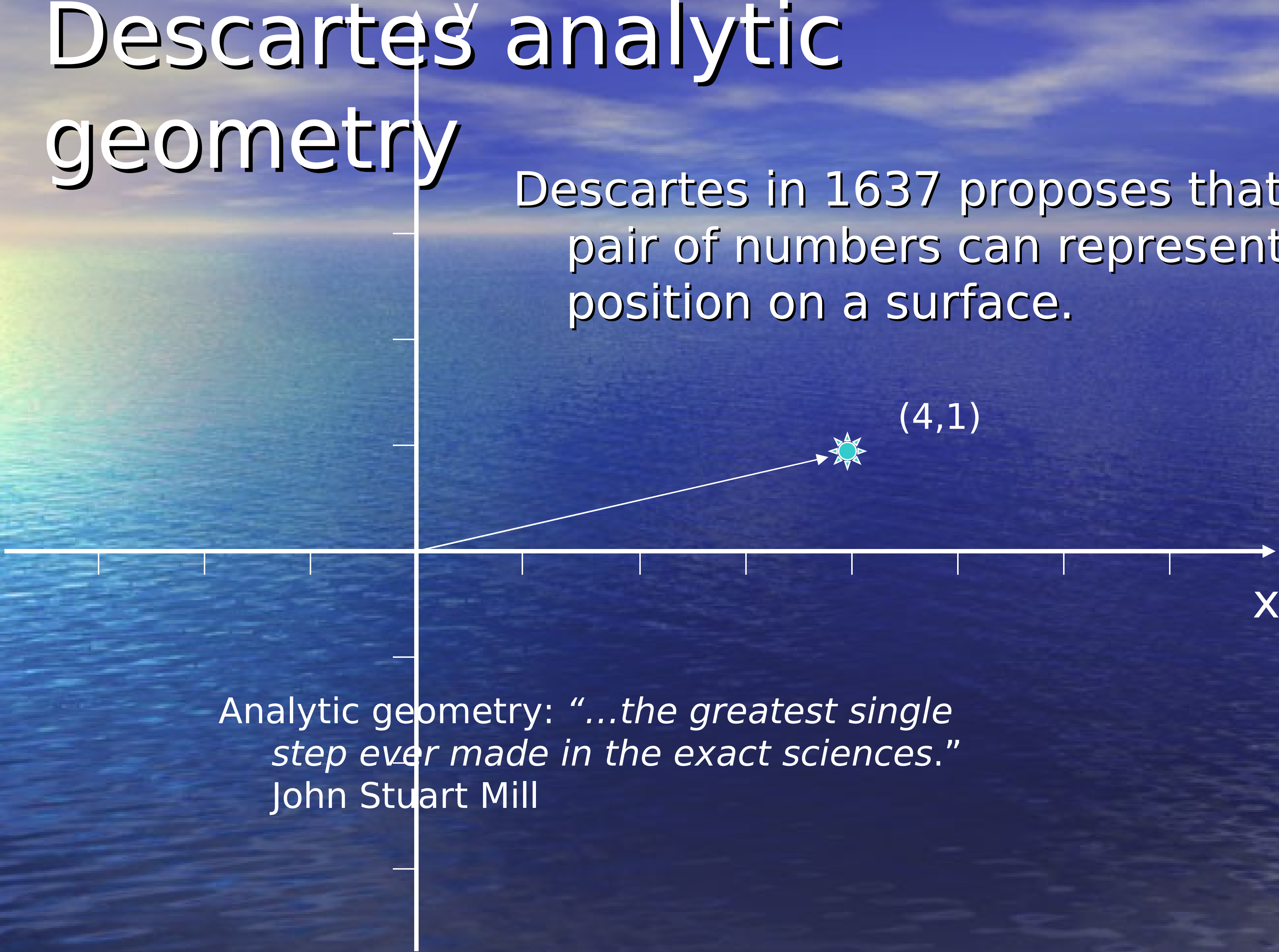
300 BC, Euclid-"Father of Geometry"

500 BC

d 547 BC, Thales-”the first true mathematician”

Descartes analytic geometry

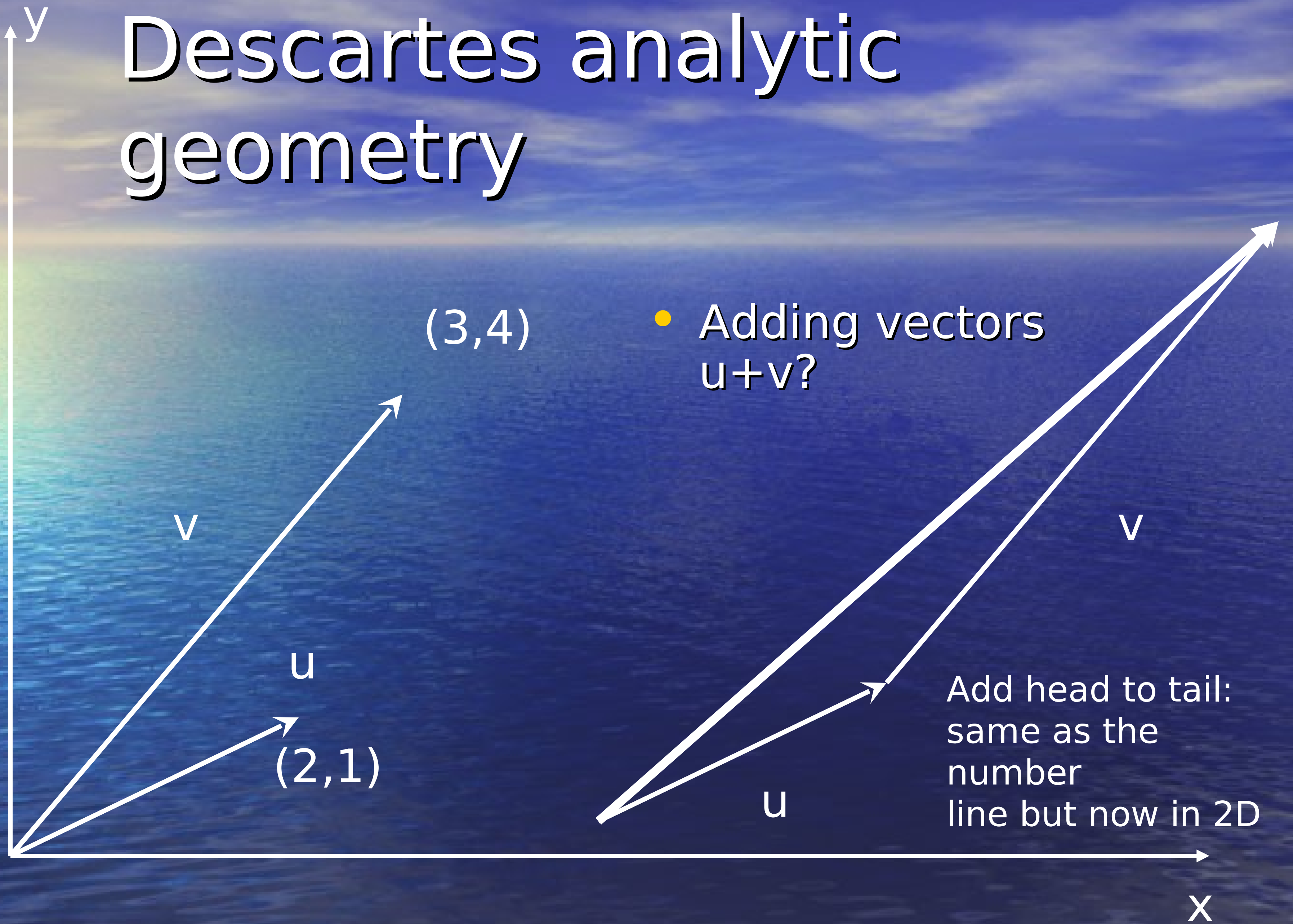
Descartes in 1637 proposes that a pair of numbers can represent a position on a surface.



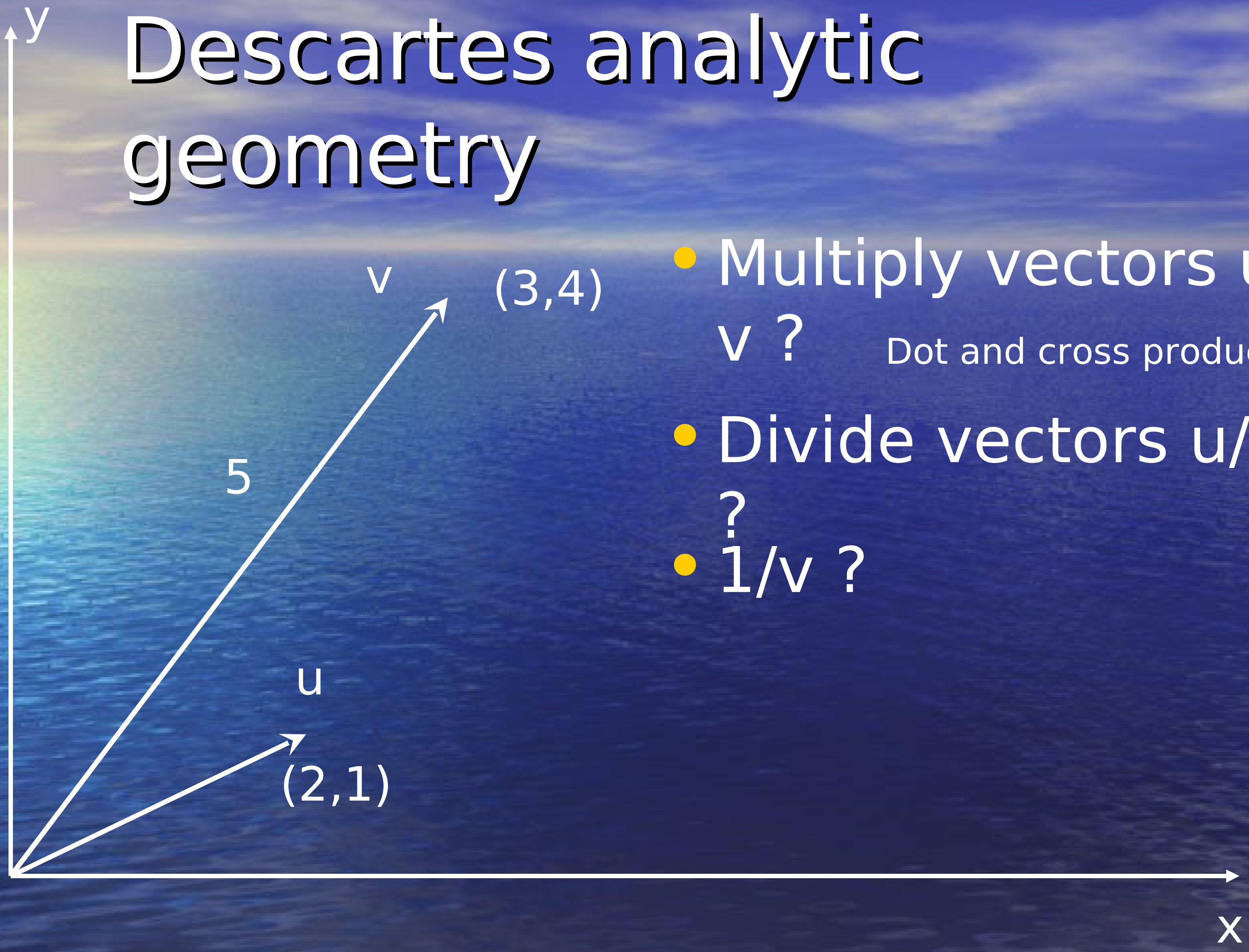
Analytic geometry: *"…the greatest single step ever made in the exact sciences*."
John Stuart Mill

Descartes analytic geometry

- Adding vectors u+v?



Descartes analytic geometry



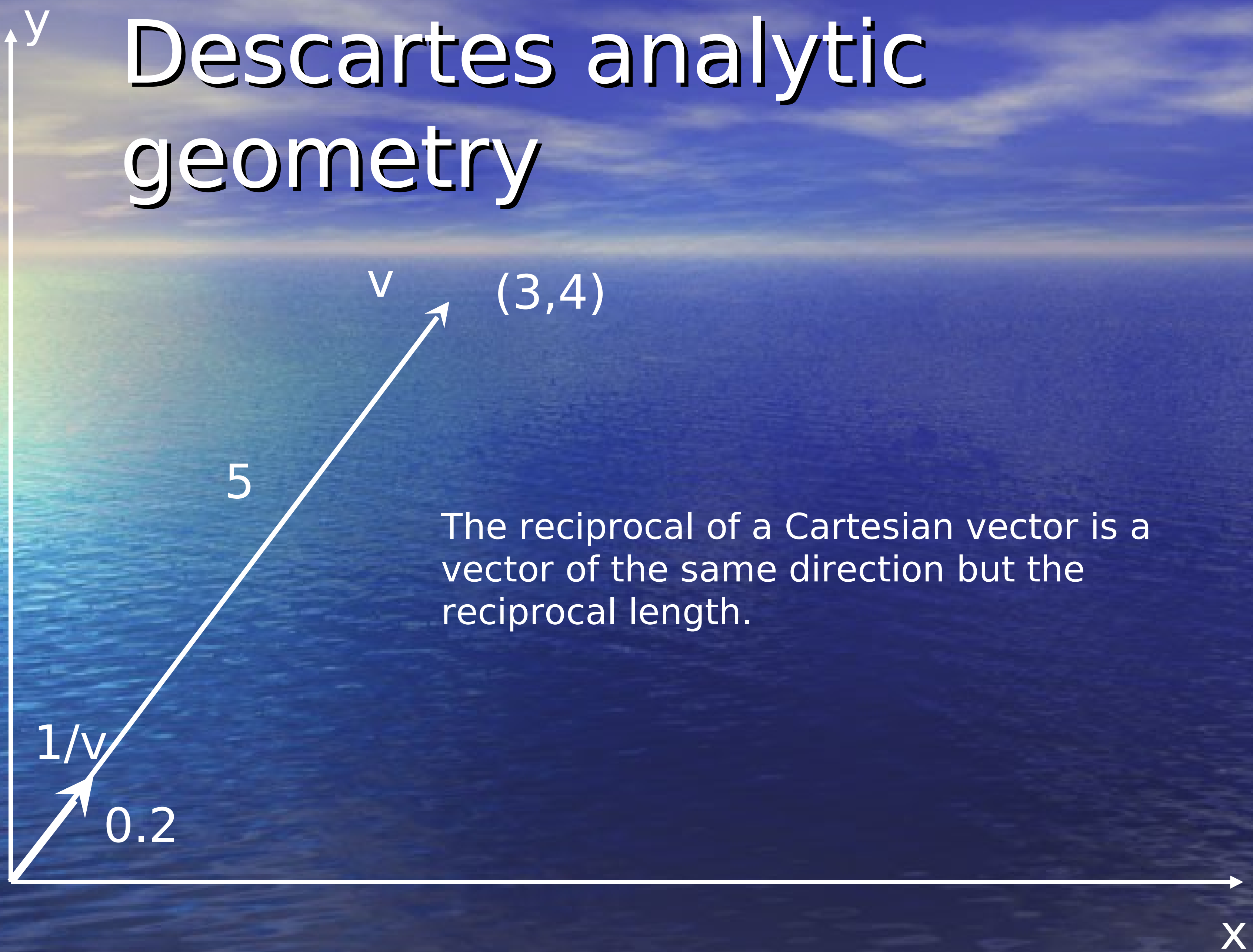
- Multiply vectors u v ?   Dot and cross products
- Divide vectors u/v ?
- 1/v ?

Descartes analytic geometry

Learning to take the reciprocal of a vector:

1. Imagine the vector lying along the number line
2. Find the reciprocal
3. Reorientate vector, bingo! Reciprocal of a vector.

Descartes analytic geometry



The reciprocal of a Cartesian vector is a vector of the same direction but the reciprocal length.

Timeline

2000

1799 Complex numbers, Argand diagram

1637 Cartesian coordinates-Descarte

1545 negative numbers established, number line

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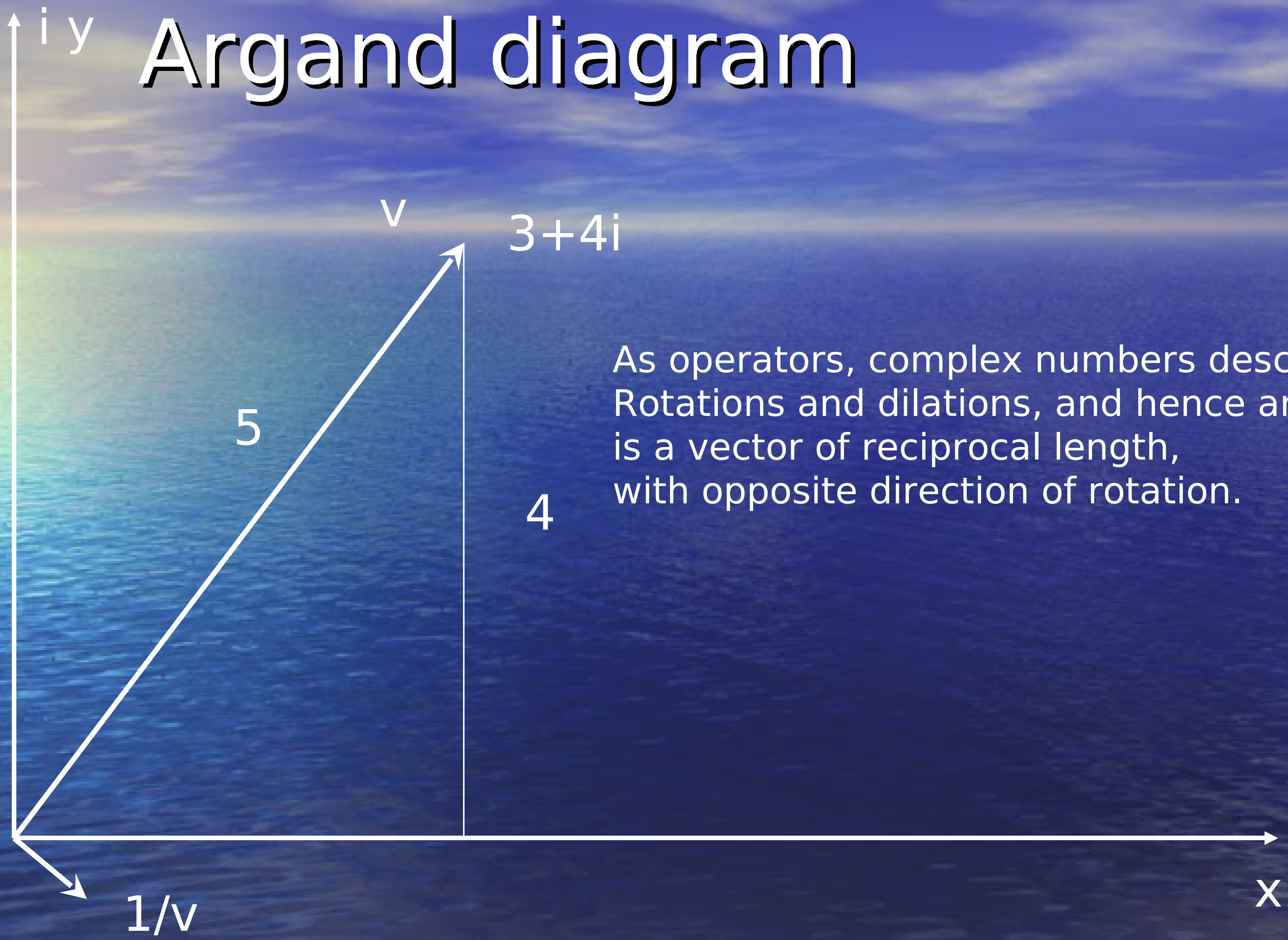
Argand diagram



As operators, complex numbers describe
Rotations and dilations, and hence an inver
is a vector of reciprocal length,
with opposite direction of rotation.

Representation vs Operator?

Timeline

What about three dimensional space?

2000

1843 Quaternions-Hamilton

1799 Complex numbers, Argand diagram

1637 Cartesian coordinates-Descarte

?!

1545 negative numbers established, number line

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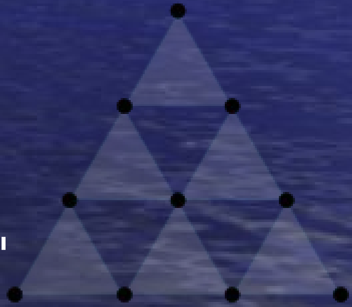
1170-1250 debts seen as negative numbers-Pisa



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j

Quaternions

- The generalization of the algebra of complex numbers to three dimensions
- $i^2 = j^2 = k^2 = -1$, $i\,j = k$,

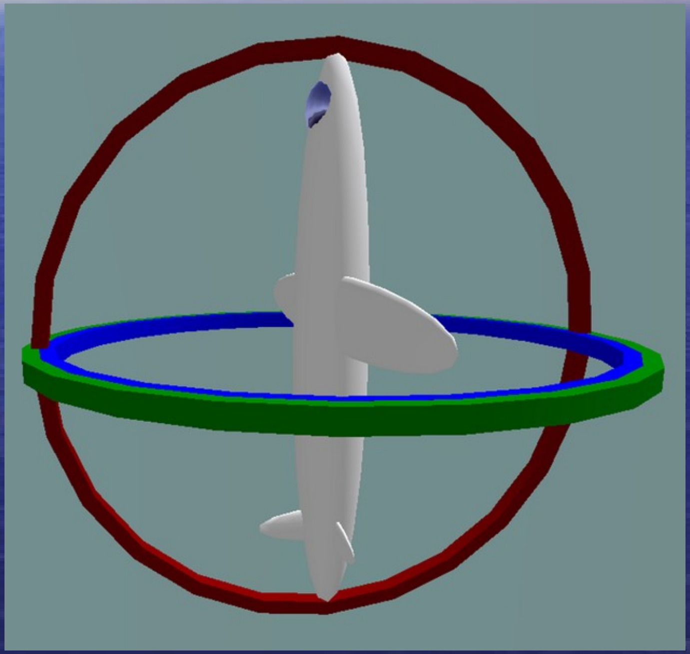
k

Non-commutative $i\,j = -j\,i$, try rotating a book

i

Use of quaternions

Used in airplane guidance systems to avoid Gimbal lock



j

Quaternions

- $i^2 = j^2 = k^2 = -1, \; i\,j = k,$

If we write a space and time coordinate as the quaternion

$$q = t + x_1 i + x_2 j + x_3 k$$

Hamilton observed this provided a natural union of space and time

k

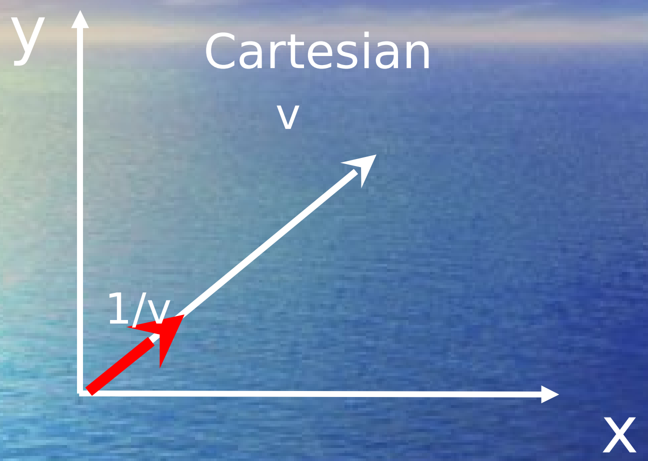
Maxwell wrote his electromagnetic equations using quaternions.

i

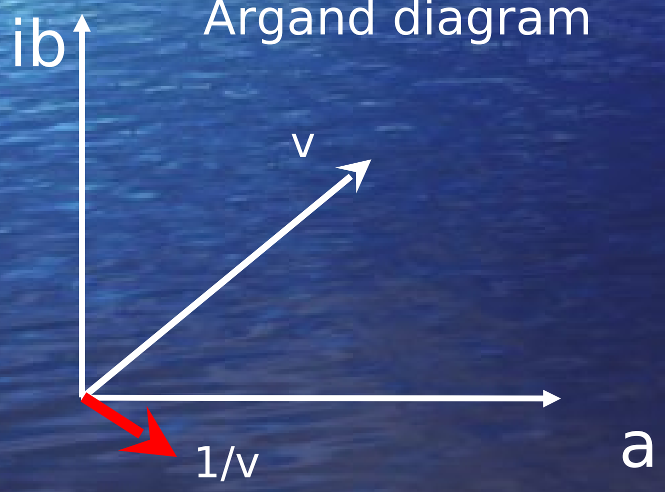
Rival coordinate systems

2D

Cartesian

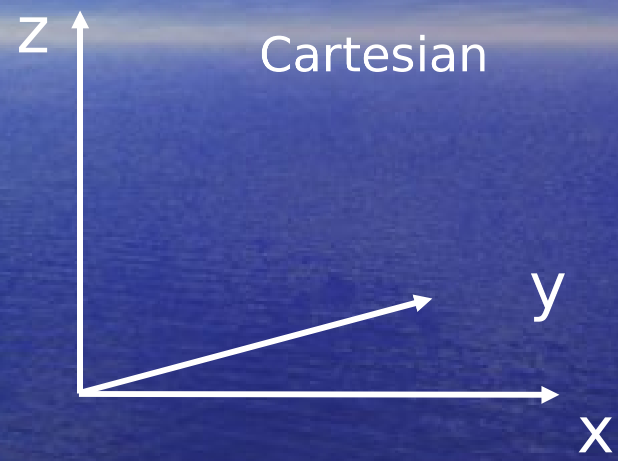


Argand diagram

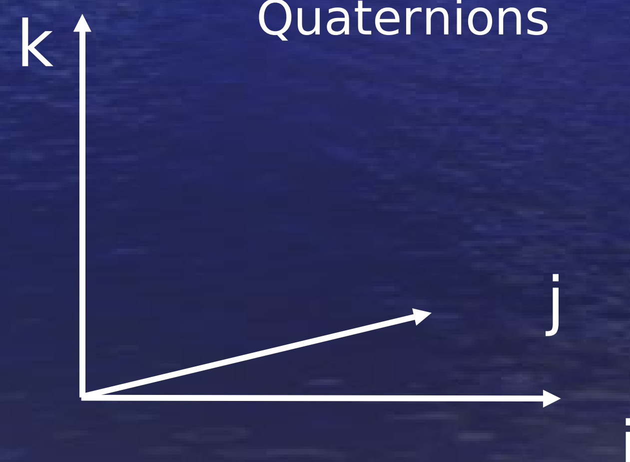


3D

Cartesian



Quaternions



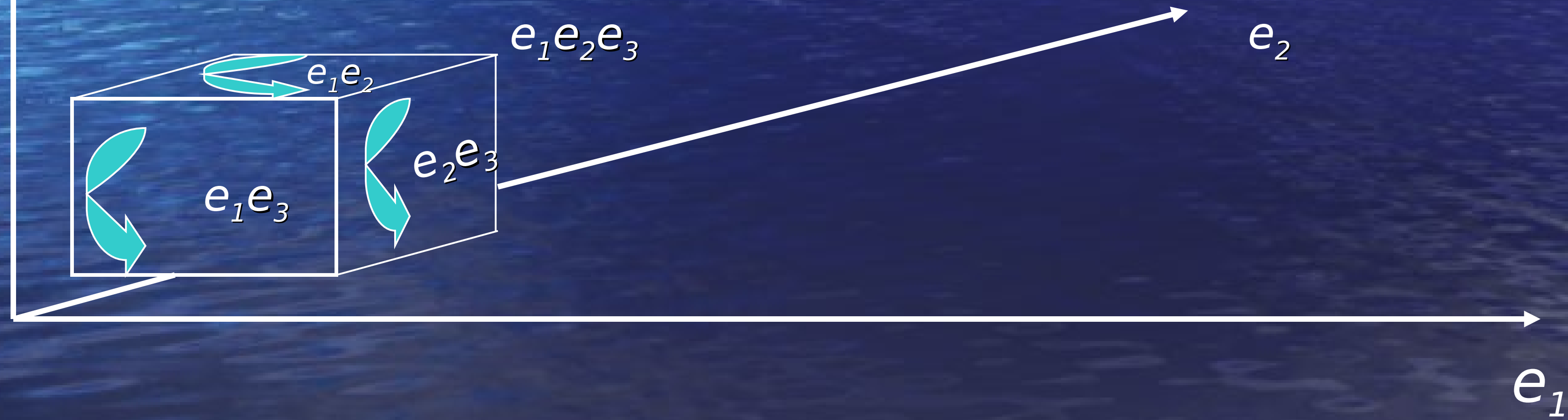
Axes non-commutative

Clifford’s Geometric Alge

- Define algebraic elements e_1, e_2, e_3
- With $e_1^2=e_2^2=e_3^2=1$, and anticommuting
- $e_i\, e_j = -\, e_j\, e_i$

This algebraic structure unifies Cartesian coordinates, quaternions and complex numbers into a single real framework.

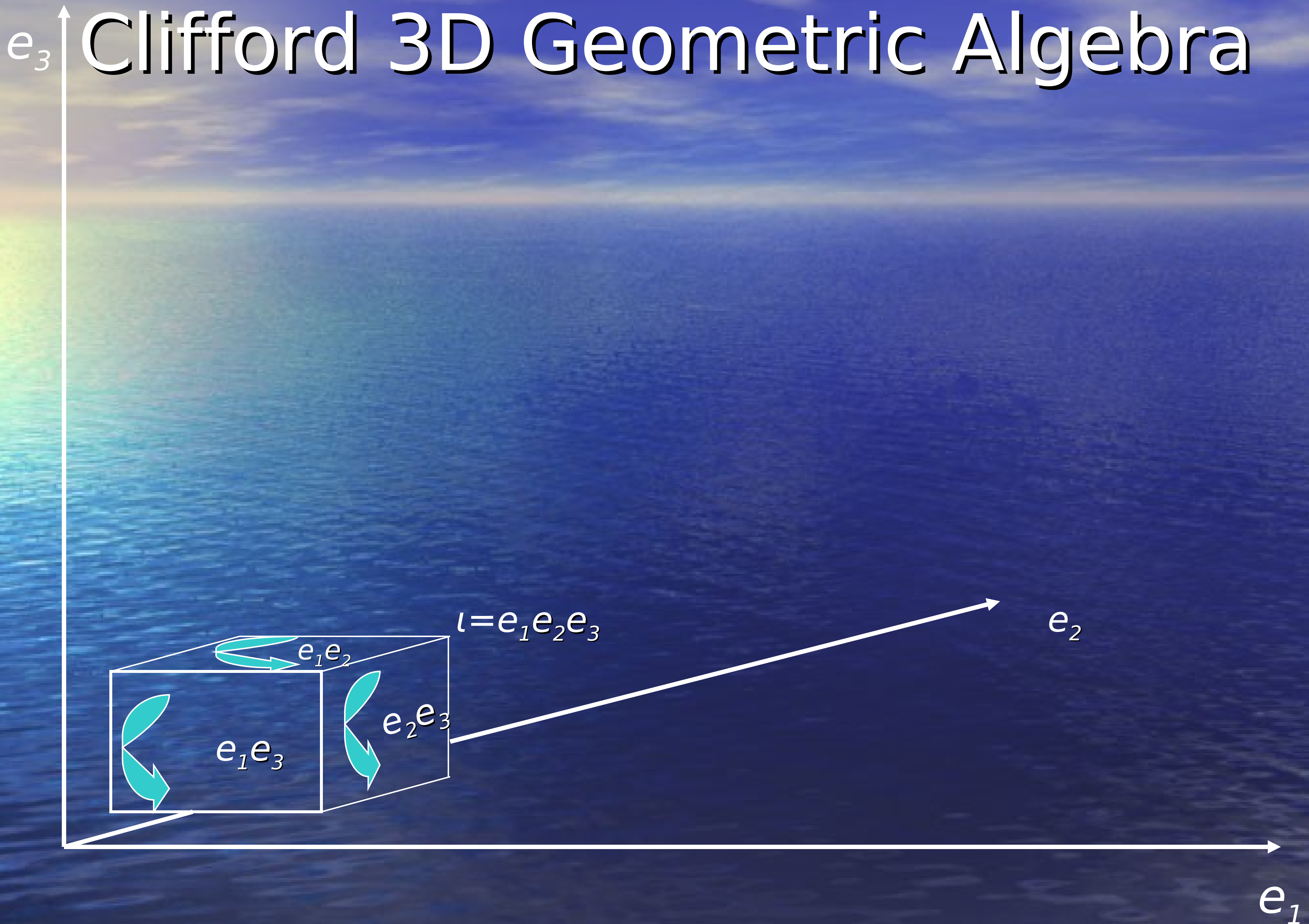
Cartesian coordinates described by e_1, e_2, e_3, quaternions by the bivectors e_1e_2, e_3e_1, e_2e_3 , and the unit imaginary by the trivector $e_1e_2e_3$.



How many space dimensions do we have?

- The existence of five regular solids implies three dimensional space(6 in 4D, 3 > 4D)
- Gravity and EM follow inverse square laws to very high precision. Orbits(Gravity and Atomic) not stable with more than 3 D.
- Tests for extra dimensions failed, must be sub-millimetre

Clifford 3D Geometric Algebra



Timeline

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1878 Geometric algebra-Clifford

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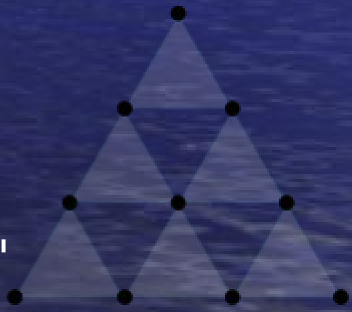

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Cliffords geometric algebra

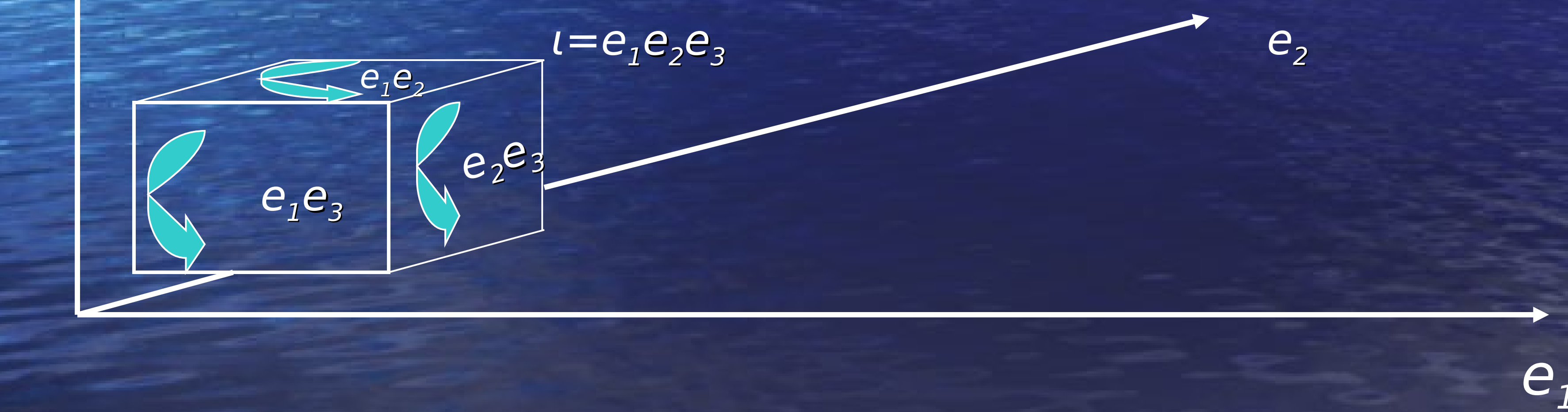
Clifford’s mathematical system incorporating 3D Cartesian coordinates, and the properties of complex numbers and quaternions into a single framework
“**should have gone on to dominate mathematical physics….”,** but….

- Clifford died young, at the age of just 33
- Vector calculus was heavily promoted by Gibbs and rapidly became popular, eclipsing Clifford’s work, which in comparison appeared strange with its non-commuting variables and bilinear transformations for rotations.

Geometric Algebra-Dual representation

$$e_2 e_3 = \iota e_1, \qquad e_3 e_1 = \iota e_2, \qquad e_1 e_2 = \iota e_3$$

$$\iota = e_1 e_2 e_3$$



The product of two vectors….

To multiply 2 vectors we….just expand the brackets…

$$
\begin{aligned}
&uv \\
&= (e_1u_1 + e_2u_2 + e_3u_3)(e_1v_1 + e_2v_2 + e_3v_3) \\
&= u_1v_1 + u_2v_2 + u_3v_3 + (u_2v_3 - v_2u_3)e_2e_3 + (u_1v_3 - v_1u_3)e_1e_3 + (u_1v_2 - v_1u_2)e_1e_2 \\
&= u_iv_i + \iota[(u_2v_3 - v_2u_3)e_1 + (u_1v_3 - v_1u_3)e_2 + (u_1v_2 - v_1u_2)e_3] \\
&= u \cdot v + \iota u \times v
\end{aligned}
$$

$$\iota = e_1e_2e_3$$

A complex-type number combining the dot and cross products!

We now note that: $u^2 = u \cdot u = u_1^2 + u_2^2$ a scalar.

Therefore the inverse vector is: $u^{-1} = u / u^2$ a vector with the same direction
and inverse length.

To check we calculate $uu^{-1} = u(u/u^2) = u^2/u^2 = 1$ as required.

Hence we now have an intuitive definition of multiplication and division of vect
subsuming the dot and cross products.

So what does $i=\sqrt{-1}$ mean?

For example: $x^2+1=0 \longrightarrow x=\pm\sqrt{-1}$

Imaginary numbers first appeared as the roots to quadratic equations. They were initially considered `imaginary', and so disregarded.

However *x* essentially represents a rotation and dilation operator. Real solutions correspond to pure dilations, and complex solutions correspond to rotations and dilations.

We can write:

$$x^2v+v=0$$

This now states that an operator *x* acting twice on a vector returns the negative of the vector. Hence *x* represents two 90deg rotations, or the bivector of the plane e_1e_2, which gives

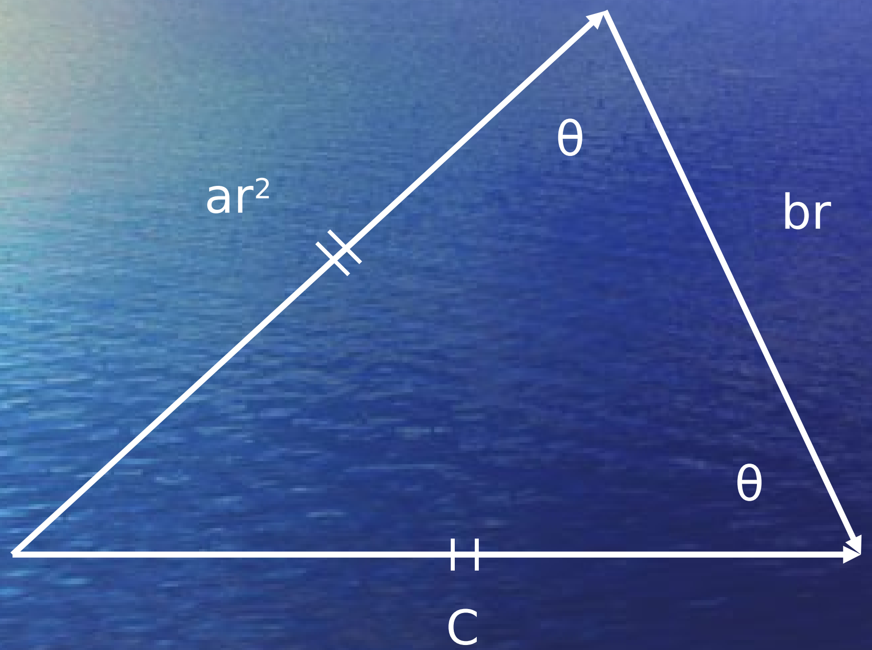
$x=\pm e_1e_2$ which implies $x^2=(e_1e_2)^2=e_1e_2e_1e_2=-1$ as required.

Hence we can replace the unit imaginary with the real geometric entity, the bivector of the plane e_1e_2.

Solving a quadratic geometrically

Solving the quadratic: $ax^2 + bx + c = 0$

is equivalent to solving the triangle:



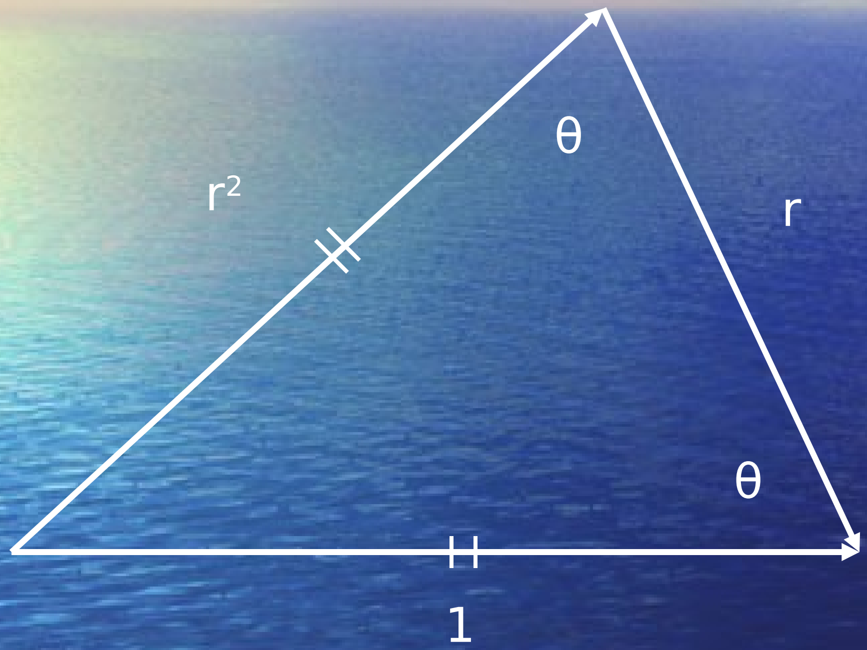
With a solution: $x = -re^{\pm\iota\theta}$ $\qquad \iota = e_1e_2$

Where *x* represents a rotation and dilation operator on a vector.

Example:

Solve the quadratic: $x^2 + x + 1 = 0$

which defines the triangle:



$$x = -re^{\pm\iota\theta} \qquad \iota = e_1 e_2$$

$$x = -e^{\pm\iota\pi/3} = -\frac{1}{2} \pm \iota\frac{\sqrt{3}}{2}$$

hus we have the two solutions, both in the field of real numbers,
th the geometric interpretation of the solutions as 60 deg rotations in the plane

Maxwell's equations

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon}, && \text{(Gauss' law);} \\
\vec{\nabla} \times \vec{B} - \frac{1}{c^2}\partial_t \vec{E} &= \mu_0 \vec{J}, && \text{(Ampère's law);} \\
\vec{\nabla} \times \vec{E} + \partial_t \vec{B} &= 0, && \text{(Faraday's law);} \\
\vec{\nabla} \cdot \vec{B} &= 0, && \text{(Gauss' law of magnetism)}
\end{aligned}
$$

Where: $\nabla = e_1\partial_x + e_2\partial_y + e_3\partial_z$

Maxwell in GA

$$uv = u \cdot v + \iota u \times v$$

$$\nabla \cdot E = \rho / \varepsilon$$

$$\nabla \times E + \partial_t B = 0$$

$$\nabla \times B - \partial_t E = \mu_0 J$$

$$\nabla \cdot B = 0$$

$$\xrightarrow{\iota}$$

$$\nabla \cdot E = \rho / \varepsilon$$

$$\iota \nabla \times E + \partial_t \iota B = 0$$

$$\iota \nabla \times \iota B + \partial_t E = -\mu_0 J$$

$$\nabla \cdot \iota B = 0$$

$$\swarrow$$

$$\nabla E + \partial_t \iota B = \frac{\rho}{\varepsilon}$$

$$\nabla \iota B + \partial_t E = -\mu_0 J$$

$$\longrightarrow \left(\partial_t + \nabla\right)\left(E + \iota B\right) = \frac{\rho}{\varepsilon} - \mu_0 J$$

Maxwell's equations in GA

$$\Box F = J$$

$\Box = \partial_t + \nabla$ Four-gradient

$F = E + \imath B$ Field variable

$J = \mu_0 \left(c\rho - J \right)$ Four-current

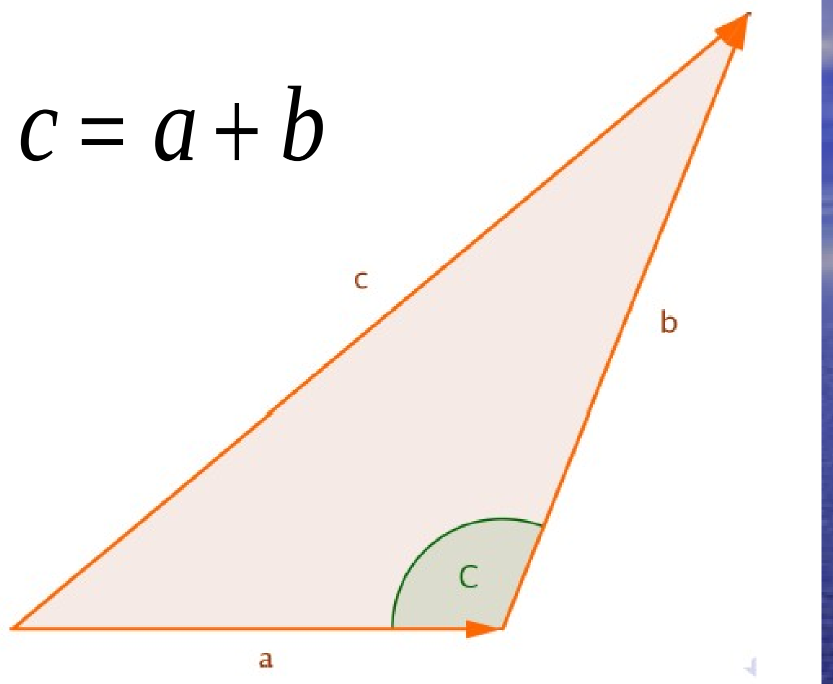
Exercise: Describe Maxwell's equations in English.

Gibb's vectors vs GA

	Gibb's vector calculus	GA
Fields	$\mathbf{E},\mathbf{B}$	$F = \mathbf{E} + \imath\mathbf{B}$
EM equations	$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon}$, $\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0$	$\Box F = J$
	$\nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mu_0 \mathbf{J}$, $\nabla \cdot \mathbf{B} = 0$	
Charge conservation	$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$	$\Box \cdot J = 0$
Energy in fields	$\frac{1}{2}\epsilon(\mathbf{E}^2 + \mathbf{B}^2)$, $\frac{1}{\mu_0 c}(\mathbf{E} \times \mathbf{B})$	$-\frac{1}{2}\epsilon_0 \lvert F \rvert^2$
Invariants	$\mathbf{B}^2 - \mathbf{E}^2$, $\mathbf{B} \cdot \mathbf{E}$	F^2
Minkowski Force	$\mathbf{K} = \gamma q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$	$K = -qvF$
Potential function	$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$, $\mathbf{B} = \nabla \times \mathbf{A}$	$F = \Box A$

Law of Cosines

$$c = a + b$$



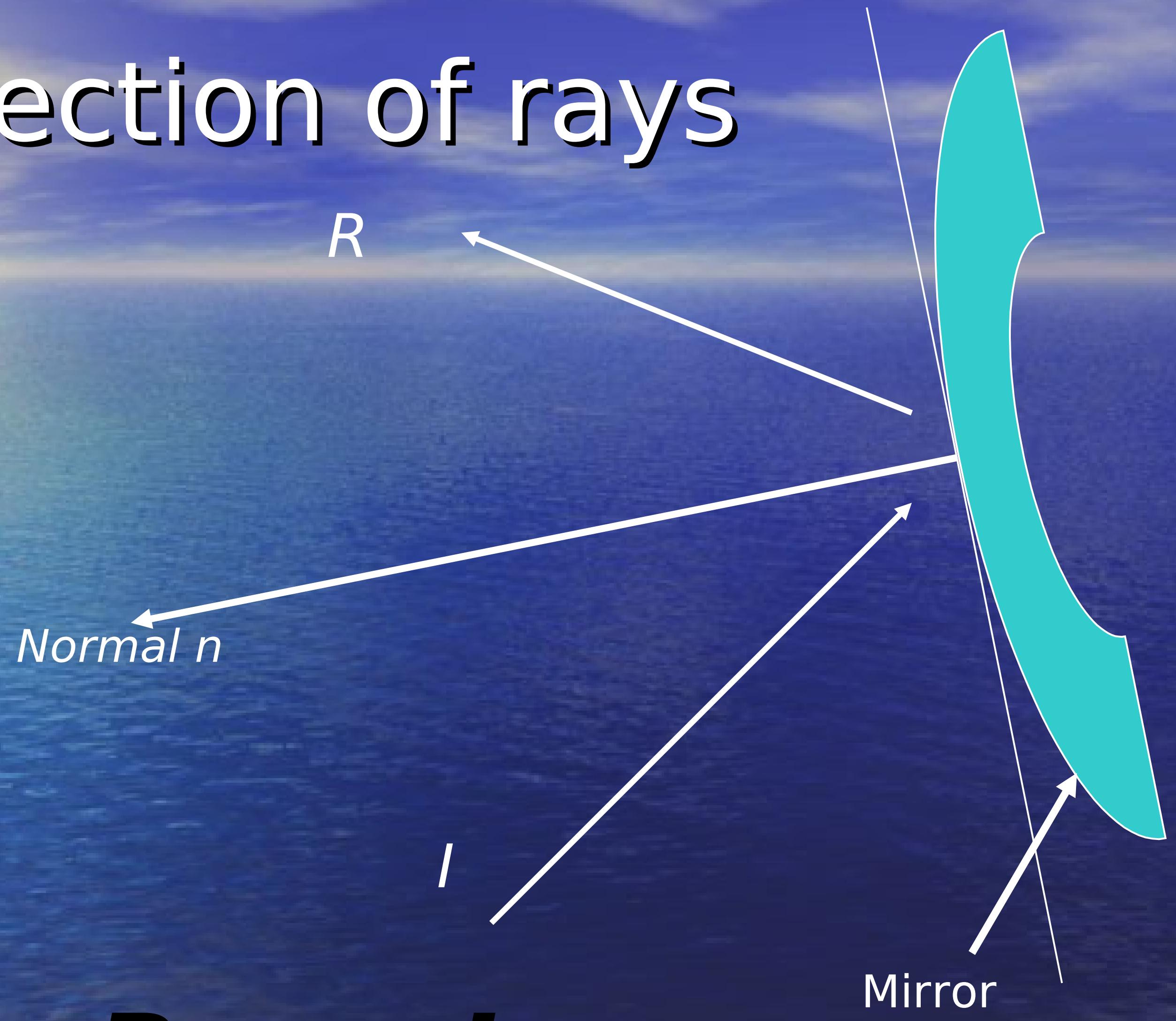
$$c^2 = (a+b)(a+b)$$

$$= a^2 + b^2 + ab + ba = a^2 + b^2 + 2a \cdot b$$

Using:

$$ab + ba = a \cdot b + \imath a \times b + b \cdot a + \imath b \times a = 2a \cdot b$$

Reflection of rays



$$\boldsymbol{R} = -\boldsymbol{n}\boldsymbol{I}\boldsymbol{n}$$

The versatile multivector (a generalized number)

$$M = a + v_1 e_1 + v_2 e_2 + v_3 e_3 + w_1 e_2 e_3 + w_2 e_3 e_1 + w_3 e_1 e_2 + b e_1 e_2 e_3$$

$$= a + v + \iota w + \iota b$$

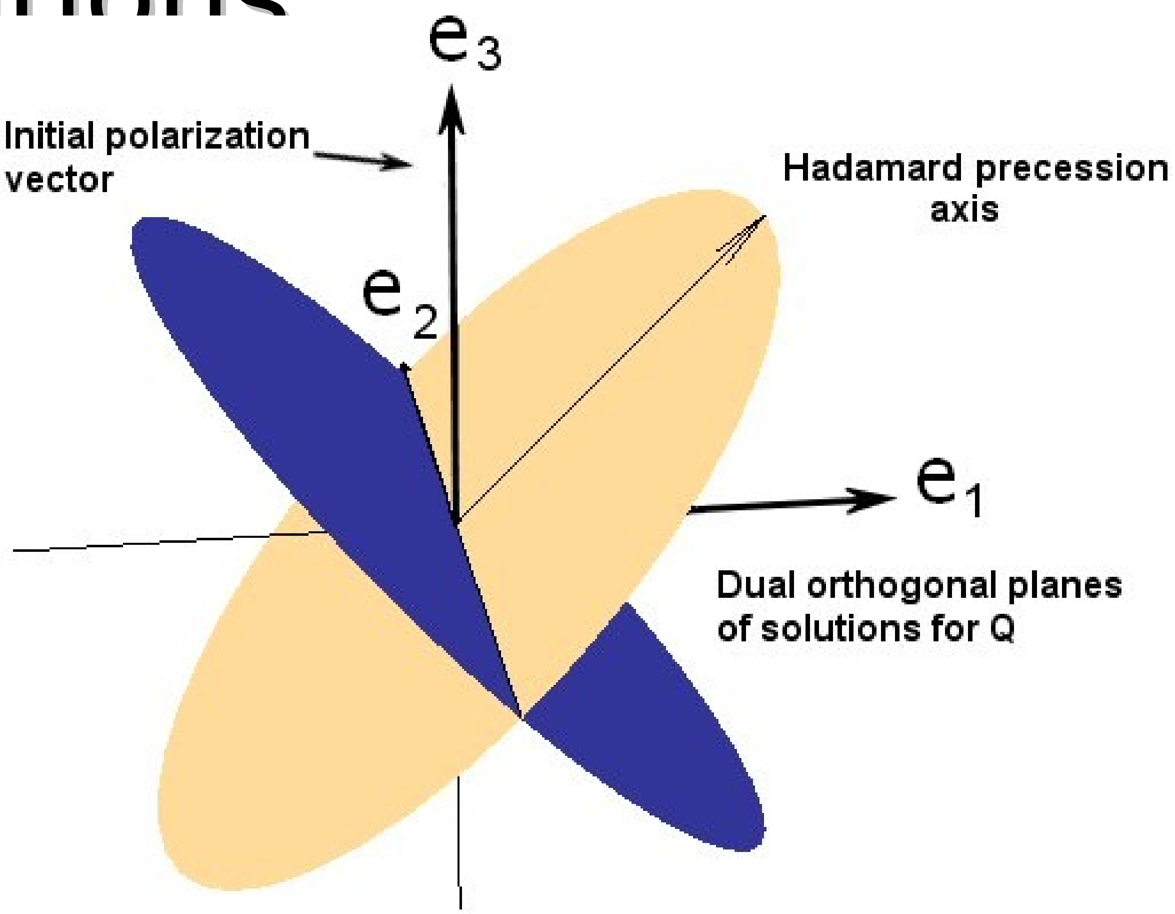
$a+\iota b$	Complex numbers
v	Vectors
ιw	Pseudovectors
ιb	Pseudoscalars
$v+\iota w$	Anti-symmetric EM field tensor $E+iB$
$a+\iota w$	Quaternions or Pauli matrices
$a+v$	Four-vectors

Research areas in GA

- black holes and cosmology
- quantum tunneling and quantum field theory
- beam dynamics and buckling
- computer vision, computer games
- quantum mechanics-EPR
- quantum game theory
- signal processing-rotations in N dimensions, wedge product also generalizes to N dimensions

$$uv = u \cdot v + i u \times v$$

Penny Flip game Qubit Solutions



Grover search in GA

$$G = -G_\sigma G_m = -(I - 2|\sigma\rangle\langle\sigma|)(I - 2|m\rangle\langle m|)$$

After $\left\lceil \frac{\pi}{4}\sqrt{\frac{N}{M}} \right\rceil$ terations of Grover operator will find solution state

In GA we can write the Grover operator as:

$$G = -\sigma m = \mathrm{e}^{\iota e_2\theta/2} e_3 e_3 \mathrm{e}^{\iota e_2\theta/2} = \mathrm{e}^{\iota e_2\theta}$$

Spinor mapping

How can we map from complex spinors to 3D GA?

$$|\psi\rangle = z_1|\uparrow\rangle + z_2|\downarrow\rangle = \begin{bmatrix} a_0 + \mathrm{i}a_3 \\ -a_2 + \mathrm{i}a_1 \end{bmatrix} \leftrightarrow \psi = a_0 + a_k \iota e_k$$

We see that spinors are rotation operators.

Probability distribution

$$P(\psi, \phi) = 2^{N-2}[\langle \psi E \psi^\dagger \phi E \phi^\dagger \rangle_0 - \langle \psi J \psi^\dagger \phi J \phi^\dagger \rangle_0]$$

Where

$$\begin{aligned} E &= \prod_{i=2}^{N} \frac{1}{2}(1 - \iota\sigma_3^1 \iota\sigma_3^i) = \frac{1}{4}(1 - \iota\sigma_3^1\iota\sigma_3^2 - \iota\sigma_3^1\iota\sigma_3^3 - \iota\sigma_3^2\iota\sigma_3^3) \\ J &= E\iota\sigma_3^1 = \frac{1}{4}(\iota\sigma_3^1 + \iota\sigma_3^2 + \iota\sigma_3^3 - \iota\sigma_3^1\iota\sigma_3^2\iota\sigma_3^3). \end{aligned}$$

Doran C, Lasenby A (2003) Geometric algebra for physicists

Conventional Dirac Equation

$$-i\hbar\gamma^\mu\partial_\mu\psi + mc\psi = 0.$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & i \\ -i & 0 \end{pmatrix}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

"Dirac has redisovered Clifford algebra..", Sommerfield

That is for Clifford basis vectors we have: $\{e_i, e_j\} = e_ie_j + e_je_i = 2e_i \cdot e_j = 2\delta_{ij}$

isomorphic to the Dirac agebra.

Dirac equation in real space

$$\Box F = m F^{*} \iota e_3$$

$$F = a + E + \iota B + \iota b$$

$$\iota = e_1 e_2 e_3$$

ame as the free Maxwell equation, except for the addition
f a mass term and expansion of the field to a full multivector

Free Maxwell equation(*J=0*): $\Box F = J$

The Maths family

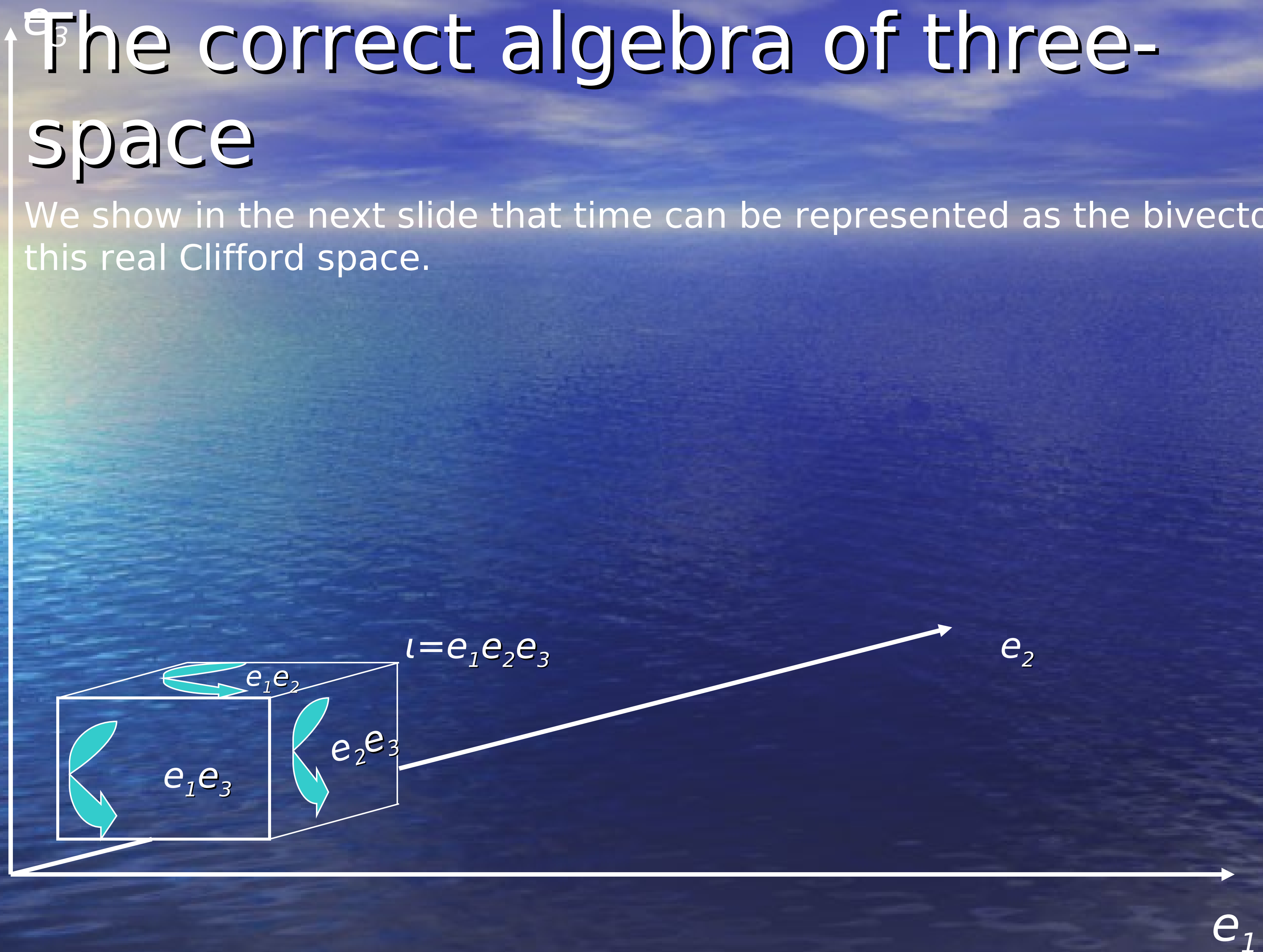
"The real numbers are the dependable breadwinner of the family, the complete ordered field we all rely on. The complex numbers are a slightly flashier but still respectable younger brother: not ordered, but algebraically complete. The quaternions, being noncommutative, are the eccentric cousin who is shunned at important family gatherings. But the octonions are the crazy old uncle nobody lets out of the attic: they are *nonassociative*."—John Baez

The multivector now puts the reals, complex numbers and quaternions all on an equal footing.

$$M = a + v + \iota w + \iota b$$

The correct algebra of three-space

We show in the next slide that time can be represented as the bivectors of this real Clifford space.



Special relativity

Its simpler to begin in 2D, which is sufficient to describe most phenomena. We define a 2D spacetime event as

$$X = x\iota + t\iota$$

So that time is represented as the bivector of the plane and so an extra Euclidean-type dimension is not required. This also implies 3D GA is sufficient describe 4D Minkowski spacetime.

We find: $X^2 = x^2 - t^2$ the correct spacetime distance.

We have the general Lorentz transformation given by:

$$X' = e^{-\phi\hat{v}/2} e^{-\iota\ \theta/2} X e^{\iota\ \theta/2} e^{\phi\hat{v}/2}$$

Consisting of a rotation and a boost, which applies uniformly to both coordinate and field multivectors.

$$P' = -\hat{\mathbf{v}} e^{-\phi\hat{\mathbf{v}}/2} P e^{\phi\hat{\mathbf{v}}/2} \hat{\mathbf{v}}$$

Compton scattering formula

Time after time

- “Of all obstacles to a thoroughly penetrating account of existence, none looms up more dismayingly than time.” Wheeler 1986
- In GA time is a bivector, ie rotation.
- Clock time and Entropy time

Foundational errors in mathematical physics

1. By not recognizing that the vector dot and cross products are two halves of a single combined geometric product.
   Circa 1910.
2. That the non-commuting properties of matrices are a clumsy substitute for Clifford’s non-commuting orthonormal axes of three-space. Circa 1930.

The leaning tower Of Pisa, Italy

u. v

u x v

Matrices as basis vectors

"...kimped a bit on the foundations,
but no one is ever going to...

Summary

- Clifford's geometric algebra provides the most natural representation of three-space, encapsulating the properties of Cartesian coordinates, complex numbers and quaternions, in a single unified formalism over the real field.
- Vectors now have a division and square root operation.
- Maxwell's four equations can be condensed into a single equation, and the complex four-dimensional Dirac equation can be written in real three dimensional space.
- SR is described within a 3D space replacing Minkowski spacetime
- GA is proposed as a unified language for physics and engineering which subsumes many other mathematical formalisms, into a single unified real formalism.

Geometric Algebra

The End

References:

http://www.mrao.cam.ac.uk/~clifford/

UNIVERSITY OF CAMBRIDGE

The geometric product magnitudes

$$
\begin{aligned}
ab &= a.b + a \wedge b \\
|a.b| &= |a||b| \cos\theta \\
|a \wedge b| &= |a||b| \sin\theta
\end{aligned}
$$

In three dimensions we have:

$$a \wedge b = \imath a \times b$$

Negative Numbers

- Interpreted financially as debts by Leonardo di Pisa,(A.D. 1170-1250)
- Recognised by Cardano in 1545 as valid solutions to cubics and quartics, along with the recognition of imaginary numbers as meaningful.
- Vieta, uses vowels for unknowns and use powers. Liebniz 1687 develops rules for symbolic manipulation.

Diophantus 200AD

$K^{Y}\overline{\alpha}\varsigma\,\overline{\iota}\ \pitchfork\ \Delta^{Y}\overline{\beta}\,\mathring{M}\,\overline{\alpha}\ \text{ἴσ}\ \mathring{M}\,\overline{\varepsilon}$ ↔ $x^3 - 2x^2 + 10x - 1 = 5$

Modern

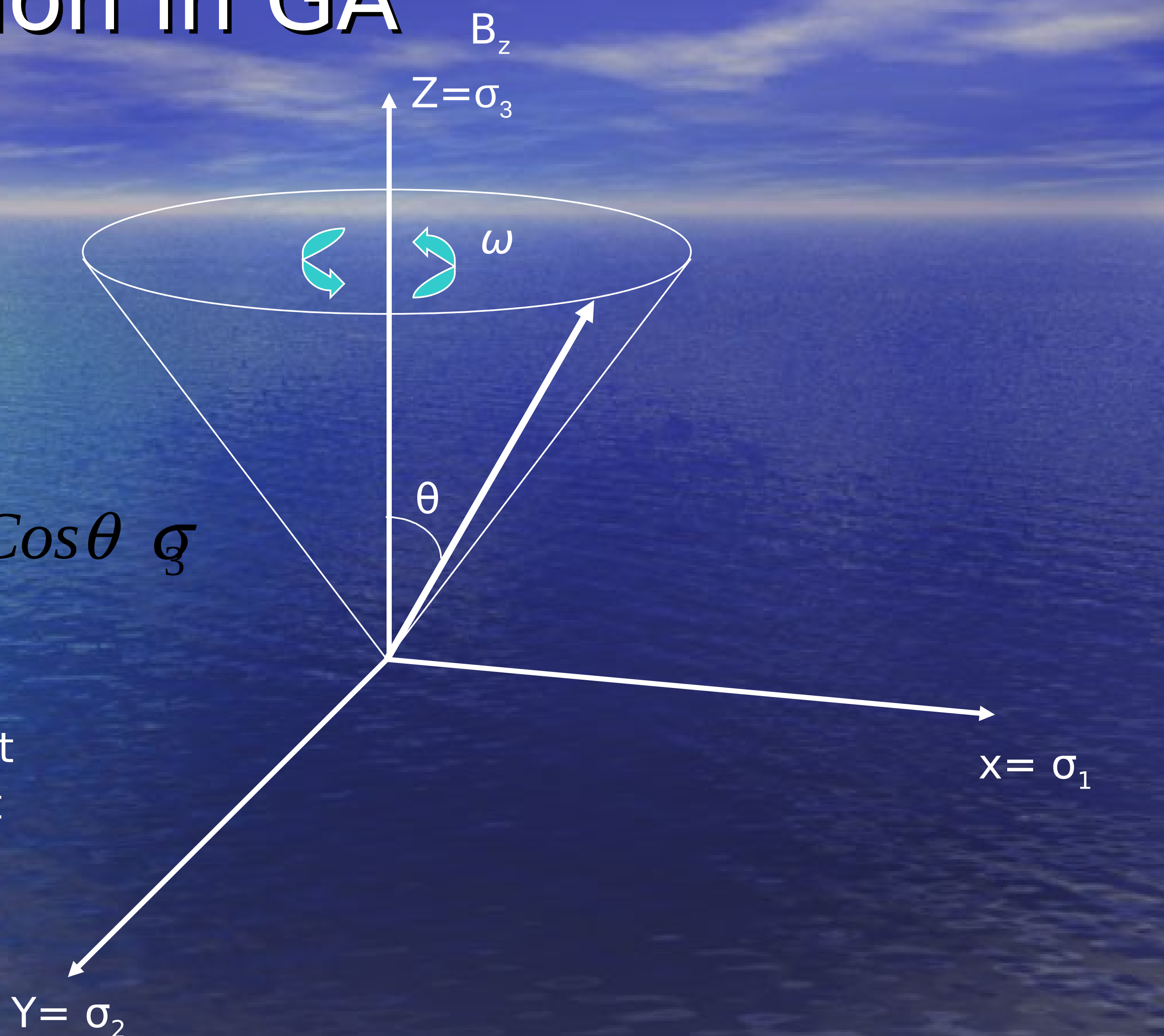
Precession in GA

Spin-1/2

$$R = e^{\omega t I z}$$

$$S = R v_0 \tilde{R}$$

$$v_0 = Sin\theta \;\sigma_1 + Cos\theta \;\sigma_3$$



$<S_x>$=Sin θ Cos ω t

$<S_y>$=Sin θ Sin ω t

$<S_z>$=Cos θ

$\omega = \gamma B_z$

Quotes

- “The reasonable man adapts himself to the world around him. The unreasonable man persists in his attempts to adapt the world to himself. Therefore, all progress depends on the unreasonable man.” George Bernard Shaw,
- Murphy’s two laws of discovery:

  “All great discoveries are made by mistake.”

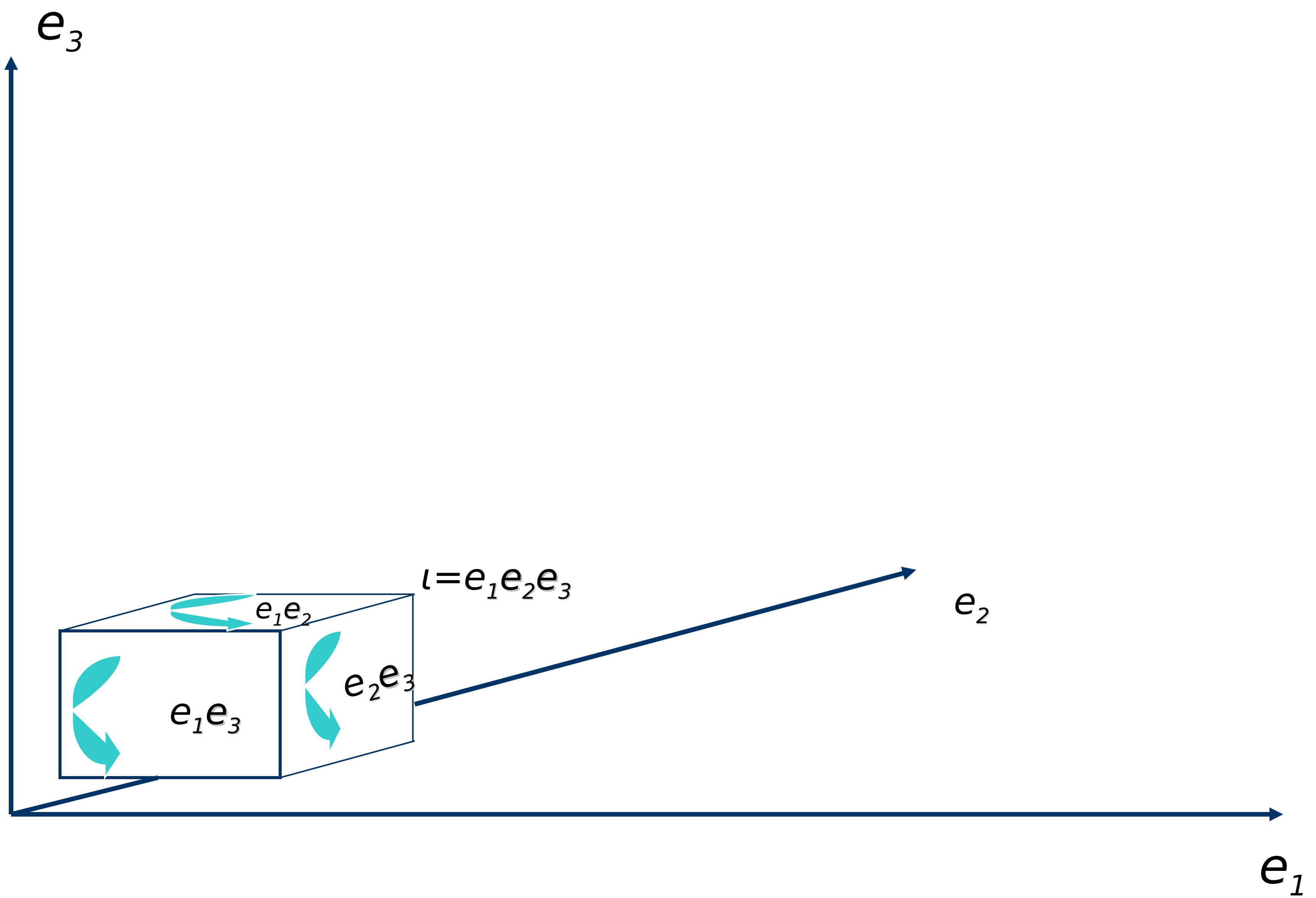
  “If you don't understand it, it's intuitively obvious.”
- “It's easy to have a complicated idea. It's very hard to have a simple idea.” Carver Mead.

Greek concept of the product Euclid Book VII(B.C. 325-265)

“1. A **unit** is that by virtue of which each of the things that exist is called one.”

“2. A **number** is a multitude composed of units.”

“16. When two numbers having multiplied one another make some number, the number so produced is called <u>plane</u>, and its <u>sides</u> are the numbers which have multiplied one another.”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